Math 255A Lecture 17 Notes

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1 Adjoint Operators and Annihilators

1.1 Translates of compact operators

Last time, we had that if $T : B \to B$ is compact, $\dim(\ker(I - T)) < \infty$.

Proposition 1.1. $\operatorname{im}(I - T)$ *is closed.*

Proof. Last time we showed that there exists a bounded sequence $x_n \in B$ such that $(I - T)x_n \to y$. We can assume that $Tx_n \to \ell \in B$, so x_n converges. In particular, $x_n \to y + \ell$. If $g = y + \ell$, then $(I - T)g = \lim_{n\to\infty}(I - T)x_n = y$. So $y \in \operatorname{im}(I - T)$. □

To show that $\dim(\operatorname{coker}(I - T)) < \infty$, we use duality arguments.

1.2 Adjoint operators

Let B_1, B_2 be Banach spaces with dual spaces B_1^*, B_2^* and the bilinear maps $B_j \times B_j^* \to \mathbb{C}$ given by $(x, \xi) \mapsto \langle x, \xi\rangle$.

Theorem 1.1. *For every $T \in \mathcal{L}(B_1, B_2)$, there exists a unique operator $T^* \in \mathcal{L}(B_2^*, B_1^*)$ such that $\langle Tx, \eta\rangle_2 = \langle x, T^*\eta\rangle_1$ for all $x \in B_1$ and $\eta \in B_2^*$. Moreover, the map $\mathcal{L}(B_1, B_2) \to \mathcal{L}(B_2^*, B_1^*)$ given by $T \mapsto T^*$ is a linear isometry.*

Proof. Let $\eta \in B_2^*$ be fixed. The map $x \mapsto \langle Tx, \eta\rangle_2$ for $x \in B_1$ is a linear continuous form on B_1 with norm $\sup_{x\neq 0} |\langle Tx, \eta\rangle_2|/\|x\| \leq \|T\|\|\eta\|$. Thus there exists a unique element $\xi \in B_1^*$ such that $\langle Tx, \eta\rangle_2 = \langle x, \xi\rangle_1$ and $\|\xi\| \leq \|T\|\|\eta\|$. The map $B_2^* \to B_1^*$ given by $\eta \mapsto \xi$ is linear and continuous of norm $\leq \|T\|$. Thus, there exists a unique operator $T^* \in \mathcal{L}(B_2^*, B_1^*)$ such that $\langle Tx, \eta\rangle_2 = \langle x, T^*\eta\rangle_1$ and $\|T^*\| \leq \|T\|$.

Now, from an earlier consequence of Hahn-Banach,

$$\|Tx\| = \sup_{\eta\neq 0} \frac{|\langle Tx, \eta\rangle_2|}{\|\eta\|} = \sup_{\eta\neq 0} \frac{|\langle x, T^*\eta\rangle_1|}{\|\eta\|} \leq \|x\|\|T^*\|.$$

So $\|T\| \leq \|T^*\|$, and the result follows. □

Definition 1.1. The operator T^* is called the **adjoint** operator of T.

1.3 Annihilators

Definition 1.2. Let B be a Banach space, and let $W \subseteq B$ be a closed subspace. The **annihilator** of W is defined as $W^o = \{\xi \in B^* : \langle x, \xi \rangle = 0 \, \forall x \in W\}$.

The annihilator is a closed subspace.

Theorem 1.2. *Let W be a closed subspace of a Banach space B.*

1. *Let $i : W \to B$ be the inclusion map. Then $i^* : B^* \to W^*$ vanishes on W^o and induces an isometric bijection $B^*/W^o \to W^*$.*

2. *Let $q : B \to B/W$ be the quotient map. Then $q^* : (B/W)^* \to B^*$ is an isometry with the range W^o.*

We have the natural isomorphisms $B^*/W^o \cong W^*$ and $(B/W)^* \cong W^o$.

Proof. The proof mainly consists of checking the definitions:

1. We have $\langle ix, \xi \rangle = \langle x, i^*\xi \rangle$ for $x \in W$ and $\xi \in B^*$. Thus, $i^*\xi$ is the restriction of ξ to W. So $\ker(i^*) = W^*$. By the Hahn-Banach theorem, every continuous linear form on W can be extended to an element of B^*. So $i^* : B^* \to W^*$ is surjective. One can check that for all $\xi \in B^*$, $\|i^*\xi\|_{W^*} = \inf_{\eta \in W^o} \|\xi + \eta\|_{B^*}$.

2. Let $q : B \to B/W$. Then $\langle qx, \eta \rangle = \langle x, q^*\eta \rangle$, where $x \in B$ and $\eta \in (B/W)^*$. Then q^* is injective, as its kernel is trivial. If $x \in W$, $0 = \langle qx, \eta \rangle = \langle x, q^*\eta \rangle$, so $\mathrm{im}(q^*) \subseteq W^o$. On the other hand, if $\xi \in W^o$, we can factor

$$B \xrightarrow{w} B/W \xrightarrow{q(x) \mapsto \langle x, \xi \rangle} \mathbb{C}.$$

So if η is the second map, then $\xi = q^*\eta$. So $\mathrm{im}(q^*) = W^o$. We can check that $\|\xi\|_{B^*} = \|\eta\|_{(B/W)^*}$. □

Theorem 1.3. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that* $\mathrm{im}(T)$ *is closed. Then* $\mathrm{im}(T^*)$ *is also closed,* $(\ker(T))^o = \mathrm{im}(T^*)$, $(\mathrm{im}(T))^o = \ker(T)^*$, $\dim(\ker(T)) = \dim(\mathrm{coker}(T^*))$, *and* $\dim(\ker(T^*)) = \dim(\mathrm{coker}(T))$.

Proof. Factorize $T = T_3 T_2 T_1$, where $T_1 : B_1 \to B_1/\ker(T)$ is the quotient map, $T_2 : B_2/\ker(T) \to \mathrm{im}(T)$ is an isomorphism, and $T_3 : \mathrm{im}(T) \to B_2$ is the inclusion map. Then $T^* = T_1^* T_2^* T_3^*$. $T_3^* : B_2^* \to (\mathrm{im}(T)) \cong B_2^*/(\mathrm{im}(T))^o$ is surjective. $T_2^* : (\mathrm{im}(T))^* \to (B_1/\ker(T)) \cong (\ker(T))^o$ is an isomorphism. $T_1^* : (\ker(T))^o \to B_1^*$ is the inclusion map. We get that $\mathrm{im}(T^*) = (\ker(T))^o$ is closed.

If $T : B \to B$, we get $(B/\mathrm{im}(T))^* \cong (\mathrm{im}(T))^o = \ker(T)$. So $\dim(\mathrm{coker}(T)) = \dim(\ker(T^*))$. The other identities can be derived similarly. □